Physics 618 2020

Finish Bloch theorem.

Stone-von-Neumann theorem.

Other Heisenberg extensions.

SvN Rep As An Induced Rep^{n}

April 24, 2020

Clean-up from last time:

1. Two separate theorems:
   - Lie's theorem
   - Cartan's classification

2. Finite-dimensional <u>complex</u> irreps of an Abelian group are one-dimensional. (Schur's Lemma.)

(A) Lie's Theorem:

Every fin. dim'l Lie algebra / $\mathbb{R}$ arises from a unique connected, simply connected Lie group $\mathfrak{g} = T_1 G$.

(B) Structure of compact Lie groups.

Reduce the problem to simple Lie groups.

Recall a group G is "simple" if it has no nontrivial normal subgroups.

A Lie <u>algebra</u> is simple if there are no nontrivial ideals.

A Lie <u>group</u> is simple if it has a simple Lie algebra.

* <u>Alt:</u> If it's connected, nonabelian, has no connected normal Lie subgroups.

A simple Lie group might not be a simple group!!

<u>Example:</u> $SU(n)$

$$Z(SU(n)) = \{\omega^j \mathbb{1} \mid \omega = e^{2\pi i/n}\}$$
$$\cong \mathbb{Z}_n.$$

Z is always a normal subgroup.

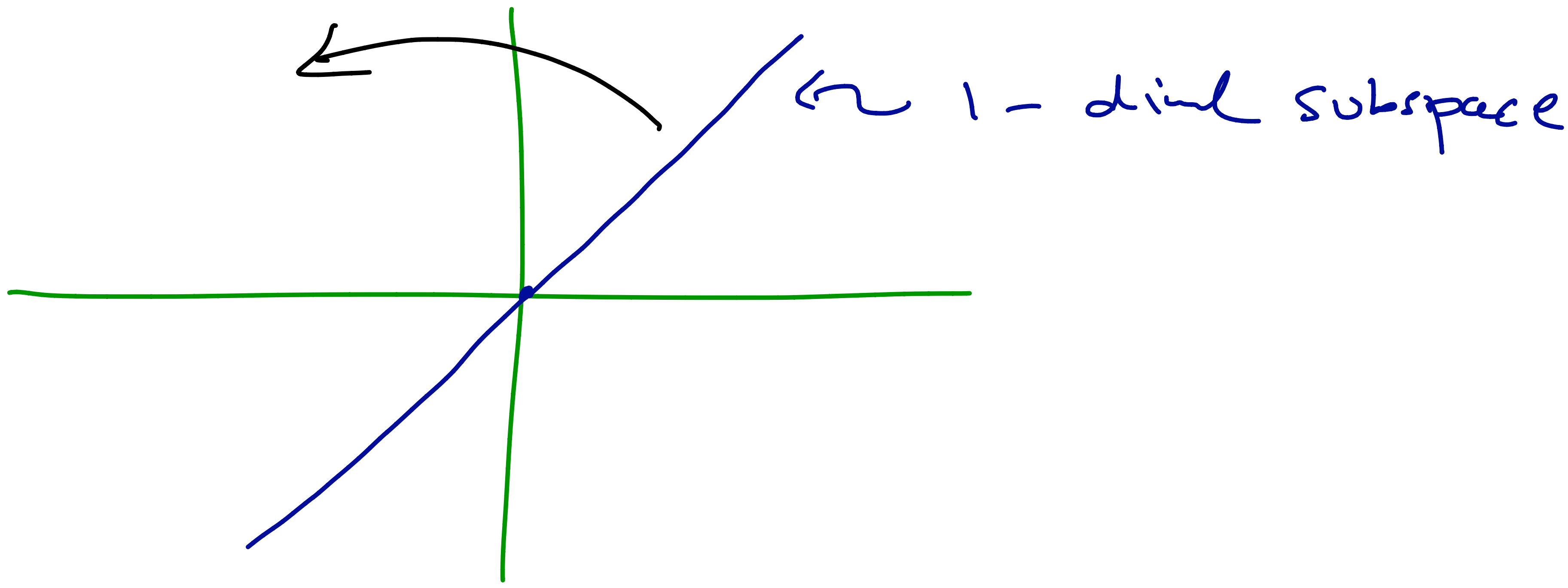
Lie algebra of $SU(n)$ = $su(n)$

$$= \left\{ n \times n \;\middle|\; \begin{array}{l} \text{traceless} \\ \text{antihermitian} \end{array} \right\}.$$

simple Lie algebra.

Irreps of Abelian groups

$G = SO(2)$ $\quad \mathbb{R}^2$ is irreducible



if we're working with real vector spaces —

$\mathbb{R}^2$ is irreducible. $\quad \mathbb{R}^2 \cong \mathbb{C}$

then it's one-dimensional.

Schur's Lemma's

$$G \qquad (V_1, \rho_1) \qquad (V_2, \rho_2)$$

a linear tmn

$$T: V_1 \longrightarrow V_2$$

is an "intertwiner," or "equivariant map"

$$T \rho_1(g) = \rho_2(g) T$$

$$\begin{tikzcd} V_1 \arrow[r, "T"] \arrow[d, "\rho_1(g)"'] \arrow[dr, phantom, "\circlearrowleft \forall g"] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "T"] & V_2 \end{tikzcd}$$

If (V_1, ρ_1) and (V_2, ρ_2) are irreducible reps.

if $T : (V_1, \rho_1) \longrightarrow (V_2, \rho_2)$

is equivariant then either

✓ a.) $T = 0$

✓ b.) T is an isomorphism so

$$(V_1, \rho_1) \approx (V_2, \rho_2)$$

<u>Proof</u>:

$$\ker T = \{ v_1 \in V_1 \mid T(v_1) = 0 \}$$

$$\operatorname{im} T = \{ v_2 \in V_2 \mid v_2 = T(v_1) \text{ for some } v_1 \}$$

Observe that $\ker T$, $\operatorname{im} T$ are invariant subspaces

e.g. $\ker T$ $\qquad T(v) = 0$

$$T(\rho_1(g) v) = \rho_2(g) T(v) = 0.$$

Because (ρ_1, V_1) (ρ_2, V_2) are irreps

$\ker T = \{0\}$ or V_1

$\operatorname{im} T = \{0\}$ or V_2 $\blacksquare$

works over <u>any</u> field.

<u>Lemma 2</u> If $T : (V, \rho) \longrightarrow (V, \rho)$ is an intertwiner and V is a complex vector space. Then

$$T = \lambda \cdot \mathbb{1} \quad \text{for some scalar } \lambda \in \mathbb{C}.$$

<u>Pf:</u> a.) Any linear op. on a f.d. <u>complex</u> vector space has at least one eigenvector.

$$\det(x - T)$$ polynomial in x must have a root in $\mathbb{C}$.

$$\det(\lambda - T) = 0.$$

b.) $\quad T v = \lambda v$

Eigenspace $E = \{ w \mid Tw = \lambda w \}$

T intertwiner then this eigenspace is an invariant subspace.

(V, ρ) irred $\Rightarrow$ $E = 0$ or $E = V$

$E \neq 0$. so $E = V$

$Tw = \lambda w \quad \forall w \in V$ $\quad \boxtimes$

Complex, fin. dim'l irreps of Abelian groups.

$T = \rho(g_0)$ this is an intertwiner:

$$T \rho(g) = \rho(g) T \quad \forall g \in G.$$

because G is Abelian.

$$T = \rho(g_0) = \lambda(g_0)\,\mathbb{1}$$

if it is irreducible must be $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ 1×1.

Γ Bloch's Theorem:

$\Gamma \circlearrowright C \subset \mathbb{R}^d$ Schröd. op.

$$H = -\frac{\nabla^2}{2m} + U(x) \qquad \boxed{U(x+\gamma) = U(x)}$$

$\rightarrow$ H commutes with $\underline{\rho(\gamma) = e^{i\gamma\cdot\hat{P}}}$

$\therefore$ Eigenspaces of H are reps of Γ

Irreps of Γ are 1-diml.

labeled by $P.D.(\Gamma) = \{\text{characters on } \Gamma\}$

$$\hat{\Gamma} \cong \mathbb{R}^d / \Gamma^\vee \ni k$$

$$\chi_{\bar{k}}(\gamma) = e^{2\pi i (k\cdot\gamma)}$$

$k \in \mathbb{R}^d$ projects to $\mathbb{R}^d/\Gamma^\vee$

$k' = k + q \quad q \in \Gamma^\vee$

wavefunctions in this 1d rep:

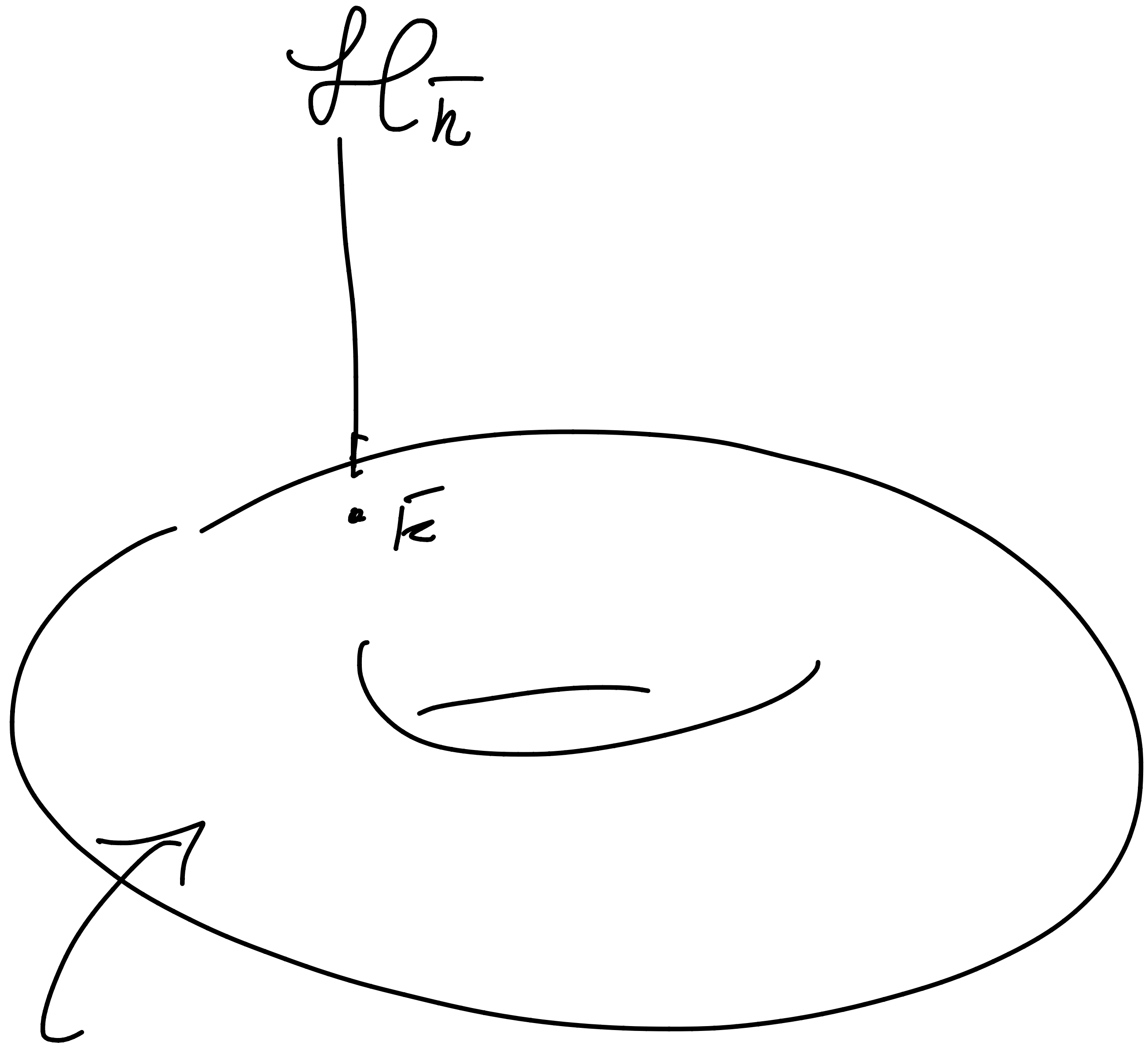
$$\psi(x+\gamma) = \chi_{\bar{k}}(\gamma)\psi(x) \quad (*)$$

$\Rightarrow \psi$ q.p. cannot be in L^2.

$H \cong \underline{\mathcal{H}_{\bar{k}}} = \left\{ \psi(x) \;\middle|\; \psi(x+\gamma) = \chi_{\bar{k}}(\gamma)\psi(x), \ \int_{\mathbb{R}^d/\Gamma} |\psi(x)|^2 dx < \infty \right\}$

$\psi : \mathbb{R}^d \to \mathbb{C}$
not a function on $\mathbb{R}^d/\Gamma$
by $(*)$ it is a section of l.b.
over torus.

Is a Hilbert space depends smoothly on $\bar{k}$



Brillouin torus $\mathbb{R}^d/\Gamma^\vee$

H acts on $\mathcal{H}_{\bar k}$

$$\psi(x+\gamma) = \chi_{\bar k}(\gamma)\,\psi(x)$$

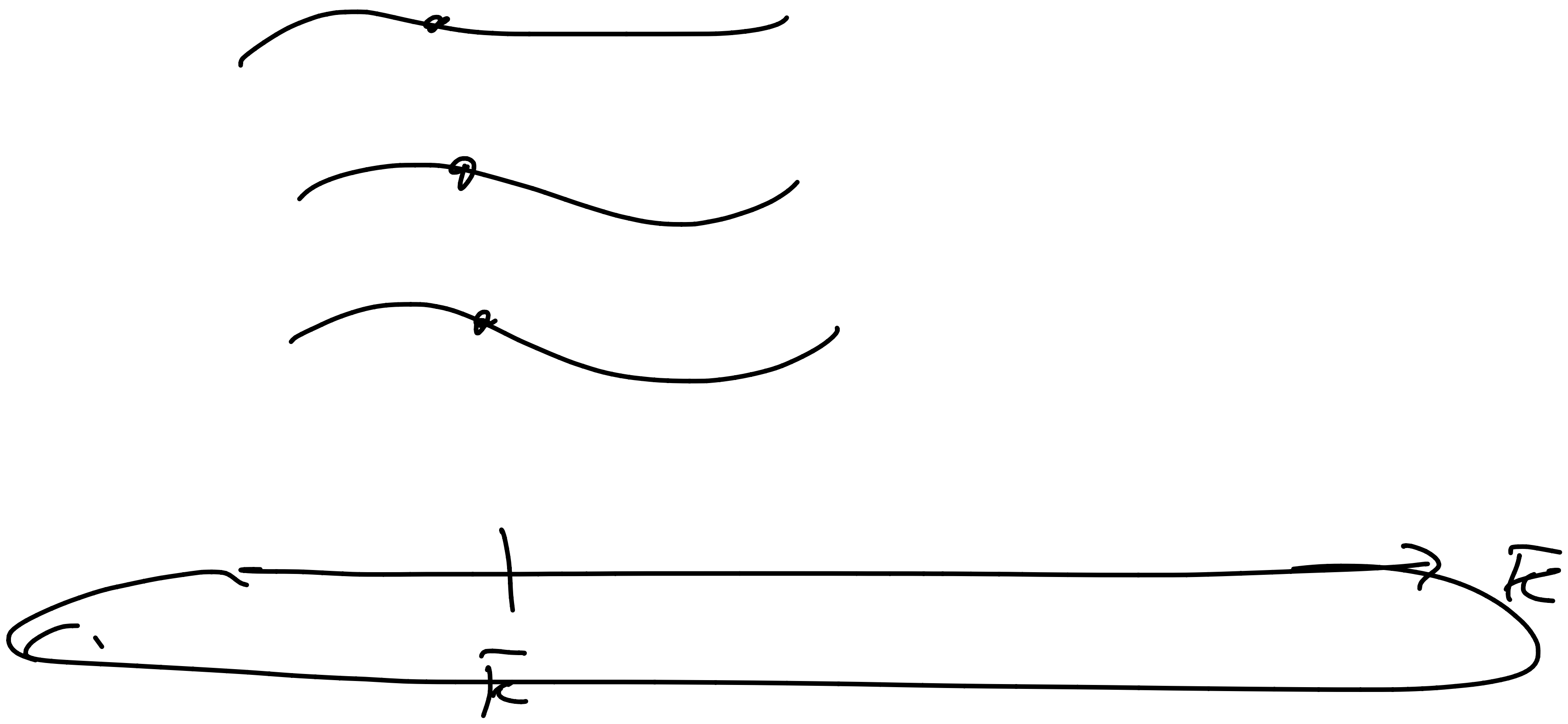
<u>Choose</u> $k \in \mathbb{R}^d$ projects to $\bar k$ $\quad$ $k' = k + q$

$$\psi(x) = e^{ik\cdot x}\,\underbrace{u(x)}$$

periodic honest function on $T = \mathbb{R}^d/\Gamma$

$$H_k = e^{-ik\cdot x} H e^{ik\cdot x}$$

$$H_{k'} = \mathcal{U} H_k \mathcal{U}^{-1}$$



$$\{E_n(k)\}_{n\in\mathbb{Z}} = \text{eigen-spectrum of } H_k$$

only depends on $\bar{k}$
and varies smoothly.

Free Electrons in $\mathbb{R}^2$ in the presence of a uniform magnetic field B

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + \cancel{V}$$

choose a gauge for $\vec{A}$ so that

$$H = \frac{1}{2m}\left[\left(p_1 + \frac{eBx_2}{2}\right)^2 + \left(p_2 - \frac{eBx_1}{2}\right)^2\right]$$

$$A_1 = -\frac{x_2}{2}B \qquad A_2 = \frac{x_1}{2}B$$

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = B.$$

Even though B is uniform, ordinary translation does <u>not</u> commute with the Hamiltonian.

We can define translation-like operators:

$$\boxed{\pi_1 = \underline{p_1} - \frac{eBx_2}{2}, \quad \pi_2 = p_2 + \frac{eBx_1}{2}}$$

$$\tilde{p}_1 = p_1 + \frac{eBx_2}{2} \qquad \tilde{p}_2 = p_2 - \frac{eBx_1}{2}$$

$$[\pi_i, \tilde{p}_j] = 0$$

$$H = \frac{1}{2m}\left(\tilde{p}_1^2 + \tilde{p}_2^2\right) \quad \text{commutes with } \pi_i$$

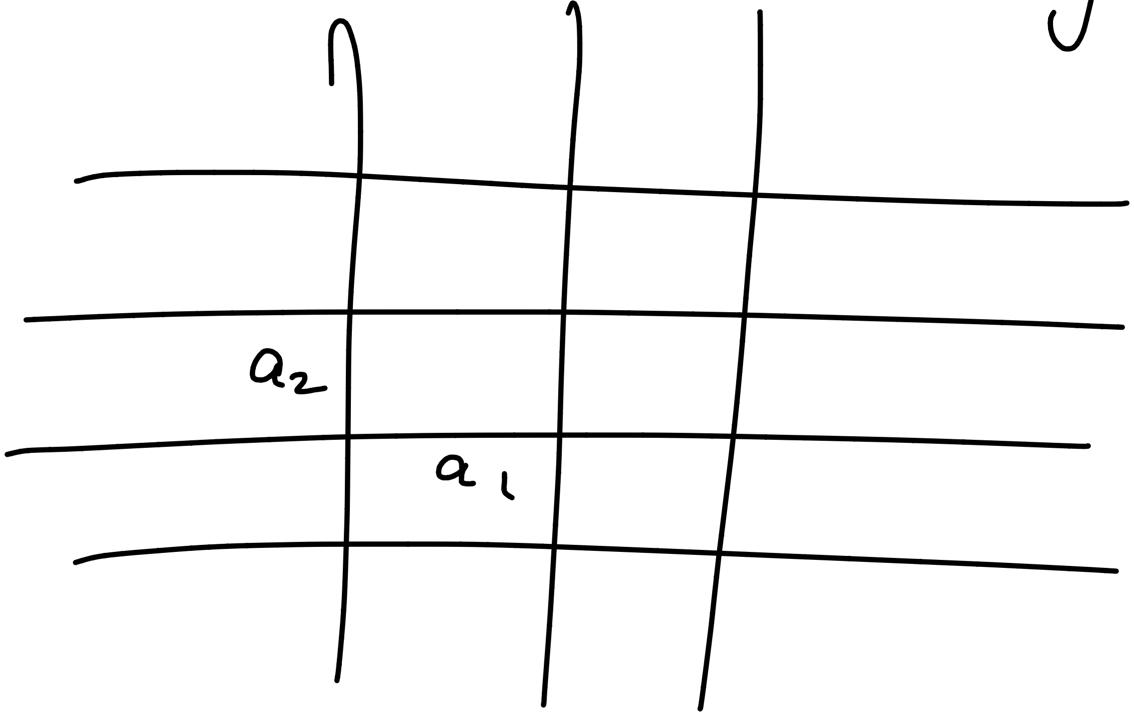
π_i : magnetic translation operators.

$$[\pi_i, H] = 0$$

$$[\pi_1, \pi_2] = - -i\hbar eB$$

"noncommuting translation operators"

(F)QHE $\longrightarrow$ "noncommutative geometry"



$$U(a_1) = \exp(i a_1 \pi_1 / \hbar)$$

$$V(a_2) = \exp(i a_2 \pi_2 / \hbar)$$

$$U(a_1) V(a_2) = \exp\left(\frac{i e B a_1 a_2}{\hbar}\right) V(a_2) U(a_1)$$

$$\exp\left(\frac{i e B a_1 a_2}{\hbar}\right) = \exp\left(2\pi i \frac{\Phi}{\Phi_0}\right)$$

$\Phi = B a_1 a_2 =$ magnetic flux through unit cell

$\Phi_0 = h/e$ "magnetic flux quantum"

$\frac{\Phi}{\Phi_0}$ irrational : irrational rotation algebra

"algebra of functions on a noncommutative torus"

Stone - von Neumann - Mackey Theorem

S - (loc. cpt.) Abelian group

$\hat{S}$ = Pontryagin dual $\{\underbrace{\chi : S \to \mathbb{C}}_{\text{characters}}\}$.

$$1 \to U(1) \to \mathrm{Heis}(S \times \hat{S}) \to S \times \hat{S} \to 1$$

$$k\big((s_1, \chi_1), (s_2, \chi_2)\big) = \frac{\chi_2(s_1)}{\chi_1(s_2)}$$

Thm guarantees $\exists$ cocycle.

$$f(\;- - - .\;) = \frac{1}{\chi_1(s_2)}$$

Natural representation of this Heisenberg group.

$$V = L^2(S)$$

$\psi_1, \psi_2 \in L^2(S)$ $\qquad \langle \psi_1, \psi_2 \rangle$

$S = \mathbb{R}^n \qquad \int_{\mathbb{R}^n} \psi_1^*(x)\psi_2(x)\, d^n x$

$S = \mathbb{Z}^n \qquad \sum_{\gamma \in \mathbb{Z}^n} \psi_1^*(\gamma)\psi_2(\gamma)$

$S = U(1) \qquad \frac{1}{2\pi}\int_0^{2\pi} (\psi_1(e^{i\theta}))^* \psi_2(e^{i\theta})\, d\theta$

similarly for $U(1)^n$

$S = \mathbb{Z}_n \qquad \frac{1}{n}\sum_{j=0}^{n-1} \psi_1^*(\omega^j)\psi_2(\omega^j)$

Let's represent $\mathrm{Heis}(S \times \hat{S})$

$$1 \to \underline{U(1)} \to \underline{\mathrm{Heis}(S \times \hat{S})} \to S \times \hat{S} \to 1$$

Try to represent the direct product $S \times \hat{S}$.

$\mathcal{H} = L^2(S) \qquad \psi : S \to \mathbb{C}$

$s_0 \in S \quad (T_{s_0} \cdot \psi)(s) = \psi(s+s_0) \quad *$

$\chi \in \hat{S} \qquad (M_\chi \cdot \psi)(s) = \chi(s)\,\psi(s) \quad *$

$$\boxed{T_{s_0} M_\chi = \chi(s_0)\, M_\chi T_{s_0}} \longleftarrow$$

$\mathcal{O}$ = group of operators generated by $T_{s_0}, M_\chi, U(1)$

$$(z;(s,\chi)) \longrightarrow z\,T_s M_\chi$$
$$\approx$$
$$\mathrm{Heis}(S \times \hat{S}) \approx \mathcal{O}.$$

Q.M. $\quad V = \mathbb{R}^n$

QFT $\quad V =$ Space of functions on spatial slice.

$$V \times V^{\vee} \longrightarrow \mathbb{R}$$

$$\chi_k(v) = e^{ik.v}$$

$$U(s)\,V(k) = e^{ik.s}\,V(k)\,U(s).$$

<u>S-vN-Mackey</u>:

Up to isomophism $\exists\,!$ irred. unitary rep^n of $\mathrm{Heis}(S \times \hat{S})$ where $U(1)$ acts by scalar mult.

$$\rho(\xi) = \xi \cdot \mathbb{1}$$

<u>Main idea of the proof</u>:

Look at a maximal Abelian Subgroup.

$$1 \to U(1) \to \underline{\mathrm{Heis}(S \times \hat{S})} \to S \times \hat{S} \to 1$$

Two obvious choices: Sequence splits over $S \times \{1_{\hat{S}}\}$ and $\{1_S\} \times \hat{S}$

Consider $\{1_S\} \times \hat{S}$

$$\left\{ \left(\underset{\substack{\cap \\ U(1)}}{1}, \ (0, \chi) \right) \right\} \quad \text{preimage in } \mathrm{Heis}(S \times \hat{S})$$

S written additively

written multiplicatively

Suppose (ρ, V) is an unitary irrep.

$$M_\chi = \rho\big((1, (0, \chi))\big)$$

$$M_{\chi_1} M_{\chi_2} = M_{\chi_2} M_{\chi_1}$$

So we can simultaneously diagonalize them.

$\exists$ ON basis ψ_α for V

$$\underbrace{M_{\chi'} M_\chi}_{M_{\chi'\chi}} \psi_\alpha = \lambda_{\alpha,\chi} \overset{M_{\chi'}}{} \psi_\alpha = \lambda_{\alpha,\chi} \lambda_{\alpha,\chi'} \psi_\alpha$$

$\chi \to \lambda_{\alpha,\chi}$ is a character on $\hat{S}$

$\therefore$ it is of the form $\chi \to \chi(s_\alpha)$ for some $s_\alpha \in S$ (Pont. duality)

So we have ψ_α basis $s_\alpha \in S$

$$M_\chi \psi_\alpha = \chi(s_\alpha) \psi_\alpha \quad \forall \alpha, \chi$$

For $s_0 \in S$ define

$$T_{s_0} = \rho((1, (s_0, 1)))$$

Now choose some α_0, ψ_{α_0}, s_{α_0}

$$M_\chi \psi_{\alpha_0} = \chi(s_{\alpha_0})\psi_{\alpha_0}$$

$$M_\chi (T_s \psi_{\alpha_0}) = \chi(s_{\alpha_0} - s)(T_s \psi_{\alpha_0})$$

Varying s gives us a copy of exactly the rep. we defined above. If (V, ρ) is irred. $\{T_s \psi_\alpha\}$ span the whole space

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A simple proof of irreducibility for $Heis(\mathbb{R}^n \times \mathbb{R}^n)$:

$v = (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$ take a section

$$s(v) = \exp(i(\alpha\hat{q} + \beta\hat{p}))$$

For $\psi_1, \psi_2 \in \mathcal{H} = L^2(\mathbb{R}^n)$ define the Wigner function on $\mathbb{R}^n \times \mathbb{R}^n$

$$W(\psi_1, \psi_2)(v) := \langle S(v)\psi_1 , \psi_2 \rangle$$

function on v determined by ψ_1, ψ_2.

Using BCH:

$$(S(v)\psi_1)(q) = e^{\frac{i\alpha\beta}{2}} e^{i\alpha q} \psi_1(q+\beta)$$

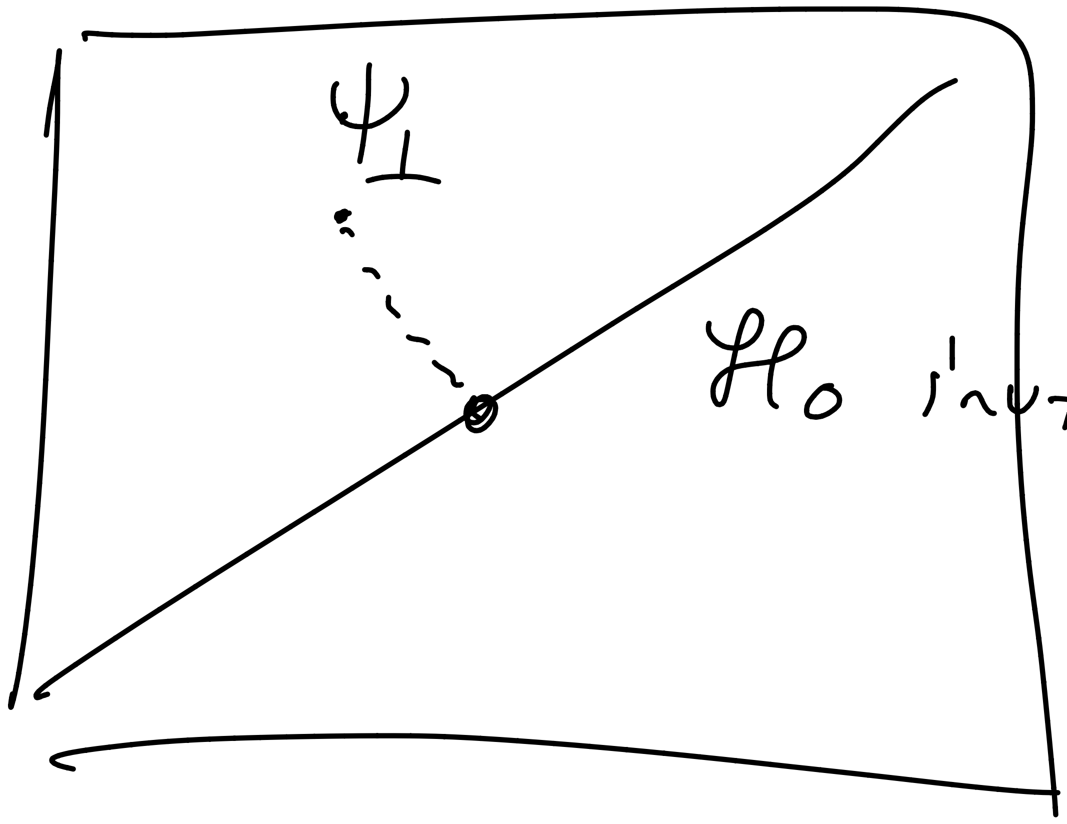
$$W(\psi_1, \psi_2)(v) = \int_{\mathbb{R}^n} e^{-i\alpha q} \psi_1^*(q+\beta/2)\psi_2(q-\beta/2) \, d^n q$$

$$\| W(\psi_1, \psi_2) \|^2 = \int | W(\psi_1,\psi_2)(v) |^2 d^{2n} v$$

$$= \|\psi_1\|^2 \|\psi_2\|^2$$

$$\|W(\psi_1,\psi_2)\|^2 = \|\psi_1\|^2\|\psi_2\|^2$$

Suppose there was an invt proper subspace $\mathcal{H}_0 \subset \mathcal{H} = L^2(\mathbb{R}^n)$



$\mathcal{H} = L^2(\mathbb{R}^n)$

$\psi_\perp \perp \mathcal{H}_0$ i.e.

$$\langle \psi, \psi_\perp \rangle = 0 \quad \forall\, \psi \in \mathcal{H}_0.$$

$$W(\underset{\mathcal{H}_0}{\underset{\cap}{\psi}}, \psi_\perp)(v) = \langle \underbrace{S(v)\psi}_{\substack{\text{vector in} \\ \mathcal{H}_0 \\ \text{for any} \\ v}}, \psi_\perp \rangle = 0.$$

$\Longrightarrow$

$$\|W(\psi, \psi_\perp)\|^2 = 0 = \|\psi\|^2 \|\psi_\perp\|^2$$

$\mathcal{H}_0 \neq 0 \quad \exists\, \psi \in \mathcal{H}_0 \quad \|\psi\|^2 \neq 0$

$\Rightarrow \|\psi_\perp\|^2 = 0 \quad \Rightarrow\Leftarrow$.

so no such $\psi_\perp$ so $\mathcal{H}$ is irreducible!